"Weak" diffusion setting

- Step 2bis (Transfer in the "weak" linear growth continuous setting): See e.g. [Jacod-Shiryaev's book 2^{nd} edition, Theorem 3.39, p.551] ([5]).

$$\widetilde{X}^{(\sigma),n} \stackrel{\mathcal{L}(\|.\|_{\sup})}{\longrightarrow} X^{(\sigma)} \quad \text{and} \quad \widetilde{X}^{(\sigma),n} \stackrel{\mathcal{L}(\|.\|_{\sup})}{\longrightarrow} X^{(\theta)} \quad \text{as } n \to +\infty.$$

- We know that, as $\sigma(t,\cdot)$ and $\theta(t,\cdot)$ have linear growth

$$\Big\| \sup_{t\in[0,T]} |\widetilde{X}^{(\sigma),n}| \Big\|_{1+\eta} + \Big\| \sup_{t\in[0,T]} |\widetilde{X}^{(\theta),n}| \Big\|_{1+\eta} \le C_{\eta,T}(1 + \|X_0\|_{1+\eta})$$

Hence, if F is $\|\cdot\|_{\sup}$-Lipschitz, then $F\big(\widetilde{X}^{(\sigma),n}\big)$, $n \ge 1$, is uniformly integrable so that

$$\mathbb{E}\, F(X^{(\sigma)}) = \lim_n \mathbb{E}\, F\big(\widetilde{X}^{(\sigma),n}\big) \qquad (\text{idem for } X^{(\theta)}).$$

- Hence $\quad \mathbb{E}\, F(X^{(\sigma)}) \le \mathbb{E}\, F(X^{(\theta)})$. □

[5] *Limit theorems for stochastic processes*, Springer, 2010.

Connection between convexity and convex ordering

- Convexity of $x \mapsto \mathbb{E}\, F(X^x)$ can be obtained as a by-product of the proof by "transferring" convexity property from discrete to continuous time...
- but also, a posteriori: in this diffusion framework

$$\boxed{\text{Convex ordering} \implies \text{Convexity}}.$$

- Let $x, y \in \mathbb{R}$, $\lambda \in [0,1]$. One has

$$\delta_{\lambda x+(1-\lambda)y} \preceq_{cvx} \lambda\delta_x + (1-\lambda)\delta_y.$$

Assume $\sigma = \theta$. Let

$X_0^{(\sigma)} = \lambda x + (1-\lambda)y$ and $\widetilde{X}_0^{(\sigma)} = \varepsilon x + (1-\varepsilon)y$, $\varepsilon \sim \mathcal{B}er(\{0,1\},\lambda) \perp\!\!\!\perp W$.

- Then $\widetilde{X}_0^{(\sigma)} \sim \lambda\delta_x + (1-\lambda)\delta_y$ and $\widetilde{X}^{(\sigma)} = \varepsilon X^x + (1-\varepsilon)X^y$ and $\mathbb{E}\,\varepsilon = \lambda$ so that, for every l.s.c. convex functional $F : \mathcal{C}([0,T],\mathbb{R}^d) \to \mathbb{R}$,

$$\mathbb{E}\, F(X^{\lambda x+(1-\lambda)y}) \le \mathbb{E}\, F(\widetilde{X}^{(\sigma)}) = \lambda \mathbb{E}\, F(X^x) + (1-\lambda)\mathbb{E}\, F(X^y).$$

- Same result for monotone convex orders (see later on).

The Euler scheme provides a simulable approximation

which preserves convex order.

Application I : Local Volatility models (functional p.c.o.c).

- New notations ($\widetilde{\sigma}$, $\widetilde{\theta}$ denote now "true" volatility)

$$dS_t = rS_t dt + S_t \widetilde{\sigma}(t, S_t) dW_t, \ S_0 = s_0 > 0,$$

where $\widetilde{\sigma} : [0, T] \times \mathbb{R} \to \mathbb{R}_+$ is a bounded continuous function.

- (At least) the weak solution exists and satisfies (see also Feller's criterion ...)

$$\widetilde{S}_t^{(\widetilde{\sigma})} := e^{-rt} S_t^{(\widetilde{\sigma})} = s_0 e^{\int_0^t \widetilde{\sigma}(s, S_s^{(\sigma)}) dB_s - \frac{1}{2} \int_0^t \widetilde{\sigma}^2(s, S_s^{(\sigma)}) ds} > 0.$$

- Idem for $\theta \rightsquigarrow \theta(t, x) = x\widetilde{\theta}(t, x)$ (same drift $x \mapsto r\,x$ of course).
- We assume that

$$0 \le \widetilde{\sigma} \le \widetilde{\kappa} \le \widetilde{\theta}$$

and $\forall\, t \in [0, T]$, $\kappa(t, \cdot) : x \mapsto x\widetilde{\kappa}(t, x)$ is convex on the whole real line.

A comparison/propagation result

Theorem (Extension of El Karoui et al. Theorem., P. 2016)

If there exists a function $\widetilde{\kappa} : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\kappa(t,\cdot) : x \mapsto x\widetilde{\kappa}(t,x) \text{ is a convex function on } \mathbb{R}$$

satisfying

(a) *Partitioning:* $0 \le \widetilde{\sigma}(t,.) \le \widetilde{\kappa}(t,.) \le \widetilde{\theta}(t,.)$ *on* $\mathbb{R}_+$, $t \in [0,T]$,

or

(b) *Dominating:* $|\widetilde{\sigma}(t,.)| \le \widetilde{\theta}(t,.) = \widetilde{\kappa}(t,\cdot)$, $t \in [0,T]$.

Then

(i) *For every every convex* $F : \mathcal{C}([0,T],\mathbb{R}_+) \to \mathbb{R}$ *with polynomial growth*

$$\mathbb{E}\, F\big(S^{(\widetilde{\sigma})}\big) \le \mathbb{E}\, F\big(S^{(\widetilde{\theta})}\big) \in (-\infty,+\infty].$$

(ii) *If* $\sigma(t,x) = x\widetilde{\sigma}(t,x)$ *is convex for every* $t \in [0,T]$, *then*

$$x \mapsto \mathbb{E}\, F\big(S^{(\widetilde{\sigma}),x}\big) \text{ is convex.}$$

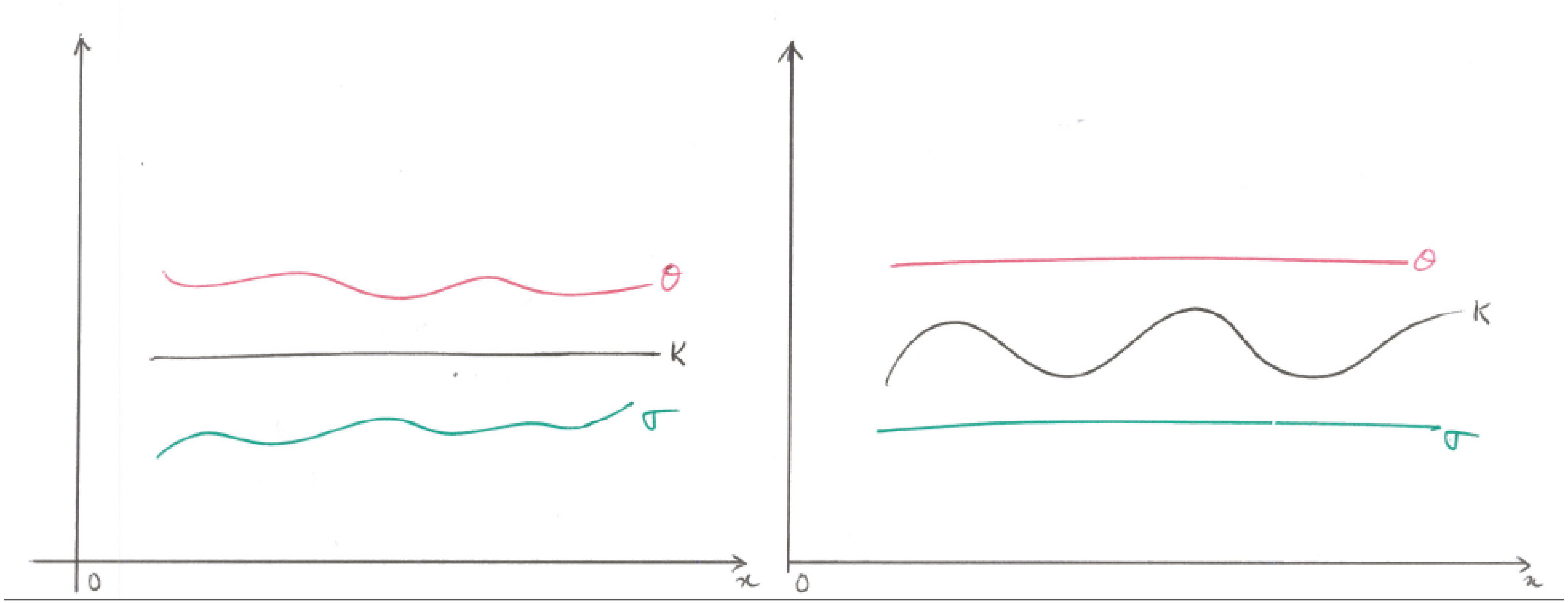


Figure: Left: flat partitioning. Right: flat bounding (El Karoui et al.).

- This theorem contains Carr et al. & Baker-Yor theorem(s).
- The method of proof applies to American style options, Lévy driven diffusions, stochastic integrals, etc (see P. 2016 and later on).
- Warning ! (1D-)Misltein scheme: does not propagate convexity.

Application II: Concave Local Vol. models (with A. Fadili)

- Concave Local Volatility (CLV) models ($\ni$ CEV): let

$\sigma(x) = x\,\widetilde{\sigma}(x) > 0$, concave $\uparrow$ on $(0, +\infty)$, $\sigma(x) = 0$, $x \le 0$, continuous.

$$dS_t = \sigma(S_t)dW_t,\ S_0 = s_0 > 0,$$

s.t. the (unique possible weak) solution satisfies $S_t \ge 0$, $t \in [0, T]$.

- Example. (Discounted) CEV model ($r = 0$) [which hits 0 a.s.. . . Ok]:

$$S_t = s_0 + \vartheta \int_0^t \sqrt{S_s}dW_s, \quad t \in [0, T].$$

- Then, for every fixed $u > 0$, the concavity property implies

$$\sigma(x) \le \Big(\sigma(u) + \sigma'(u)(x - u)\Big)_+,\ x \in \mathbb{R}_+$$

so that, if we set

$$dX_t^{(u)} = \Big(\sigma(u) + \sigma'(u)(X_t^{(u)} - u)\Big)_+ dW_t,\ X_0^{(u)} = s_0$$

then, for every convex vanilla payoff $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$

$$\boxed{\mathbb{E}\,\varphi(S_T) \le \inf_{u>0} \mathbb{E}\,\varphi\big(X_T^{(u)}\big)}$$

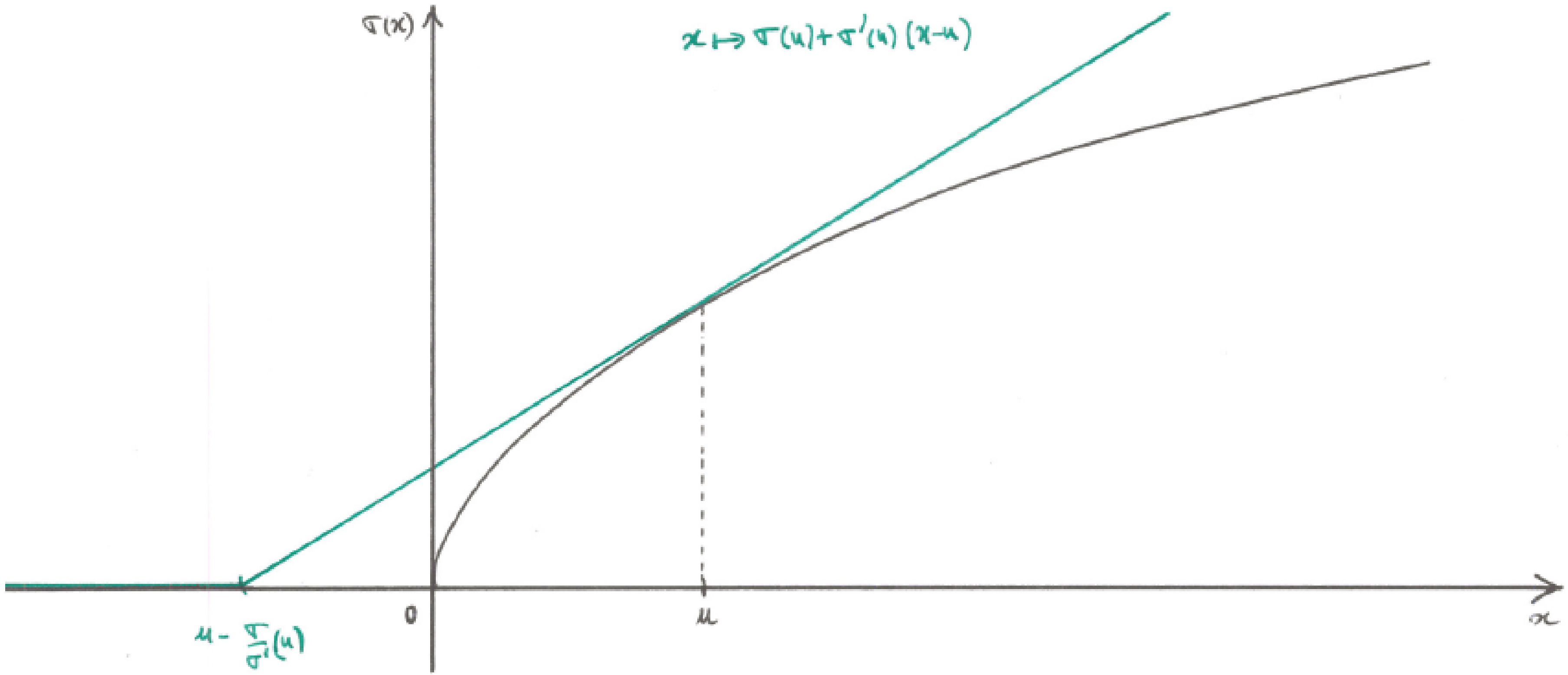


Figure: Black-Scholes convex domination of a Local Volatility model.

Back to Black(-Scholes)

- Set $\xi(u) := \frac{\sigma'(u)}{\sigma(u)} - u > 0$ (by concavity). Hence

$$X_t^{(u)} + \xi(u) = \underbrace{s_0 + \xi(u)}_{>0} + \int_0^t \underbrace{\sigma'(u)}_{\geq 0} \left(X_s^{(u)} + \xi(u)\right)^+ dW_s.$$

By strong uniqueness, $X_t^{(u)} + \xi(u) = Y_t^{(u)}$ where $Y^{(u)}$ satisfies Black-Scholes dynamics

$$Y_t^{(u)} = Y_0^{(u)} + \sigma'(u) \int_0^t Y_s^{(u)} dW_s.$$

- Example: if $\varphi(x) = (x - K)_+$ is a vanilla Call payoff

$$\mathbb{E}\,(S_T - K)_+ \leq \inf_{u>0} \mathrm{Call}_{BS}\Big(s_0 + \xi(u), K + \xi(u), \sigma'(u), 0, T\Big).$$

Proposition (Tractable upper-bound)

One has

(*i*) $u \mapsto \mathbb{E}\,(Y_T^{(u)} - K)_+$ *is differentiable and* $\frac{\partial}{\partial u}\mathbb{E}\,(Y_T^{(u)} - K)_+ \geq 0$ *on* $\big[\max(s_0, K), +\infty\big)$

(*ii*) *Hence*

$$\mathbb{E}\,(S_T - K)_+ \leq \min_{0 \leq u \leq \max(s_0, K)} \mathrm{Call}_{BS}\big(s_0 + \xi(u), K + \xi(u), \sigma'(u)\big)$$

leading to a faster search for the argmin.

Practitioner's corner: – In fact $u_{\min}$ lies not far from s_0 and K.

– Exploration starting from $\frac{s_0+K}{2}$.

Back to $1D$-models

Question: is convexity of σ always mandatory (e.g. in one dimension)?

At least for some specific functionals ?

Is convexity necessary ? $\sigma(t,x)=\sigma(x)$, $d=q=1$

- We assume for a while that $d=q=1$.
- One shows [Jourdain-P. '23] that (when $d=1$)

$$\sqrt{\frac{2}{\pi}}|\sigma(x)| = \lim_{t\to 0}\frac{1}{\sqrt{t}}\mathbb{E}|X_t^x - x| = \lim_{t\to 0}\frac{1}{\sqrt{t}}\mathbb{E}|X_t^x - X_0^x| = \lim_{t\to 0}\frac{1}{\sqrt{t}}\mathbb{E}F_t(X^x)$$

  with $F_t(\alpha)=|\alpha(t)-\alpha(0)|$ an (only) 2-marginal convex functionals.
- As soon as convexity propagation for 2-marginal functionals holds true then $|\sigma|$ is convex !!
- **Conclusion:** The convexity assumption on either σ or ϑ is mandatory ...except, maybe, for 1-marginal convex order when $d=q=1$.

A discrete time counterexample: back to ARCH

- Still with

$$X^x_{k+1} = X^x_k + \sigma_k(X^x_k)Z_{k+1},\ k = 0, n-1,\ X_0 = x \in L^1(\mathbb{P}).$$

- Assume that, for every (Lipschitz) convex function $F : \mathbb{R}^2 \to \mathbb{R}$, $x \mapsto \mathbb{E}\, F(X_k, X_\ell)$ is convex.
- Then $x \mapsto \mathbb{E}|X^x_1 - x|$ is convex i.e.

$$x \longmapsto \mathbb{E}|\sigma_0(x)Z_1| = |\sigma_0(x)|\mathbb{E}\,|Z_1| \quad \text{is convex.}$$

- Hence $x \mapsto |\sigma_0(x)|$ is convex.

1-marginal of $1D$ diffusion (after [El Karoui et al.]) Direct approach

- Back to standard 1D martingale diffusion where $\sigma \in \mathcal{C}^{0,1}_{lin_x, Unif_t}([0,T]\times\mathbb{R})$

$$dX_t^x = \sigma(t, X_t^x)dW_t, \quad X_0^x = x \in \mathbb{R}.$$

- If ([6]) f is smooth then $\partial_x \mathbb{E}\, f(X_T^x) = \mathbb{E}\, f'(X_T^x) Y_T^{(x)}$, where $Y^{(x)}$ is the tangent process:

$$Y_t^{(x)} = \mathcal{E}\Big(\int_0^{\cdot} \sigma'_x(s, X_s^x)dW_s\Big)_t := \exp\Big(\int_0^t \sigma'_x(s, X_s^x)dW_s - \tfrac{1}{2}\int_0^t \sigma'_x(s, X_s^x)^2 ds\Big).$$

- Let $\mathbb{Q} = Y_T^{(x)} \cdot \mathbb{P}$, the probability on $(\Omega, \mathcal{A}, \mathbb{P})$ under which (Girsanov)

$$B_t = W_t - \int_0^t \sigma'_x(s, X_s^x)ds \quad \text{is a standard } \mathbb{Q}\text{-Brownian motion.}$$

- Then

$$X_t^x = x + \int_0^t \sigma\sigma'_x(s, X_s^x)ds + \int_0^t \sigma(s, X_s)dB_s$$

and

$$\partial_x \mathbb{E}\, f(X_T^x) = \mathbb{E}_{\mathbb{Q}}\, f'(X_T^x).$$

[6] see El Karoui et al. 1998, Robustness of the Black and Scholes formula, *Math. Fin.*

Direct approach: conclusion ($d = 1$)

- If $\sigma\sigma'_x$ is Lipschitz in space uniformly in time, then ([7]).

$$\mathbb{Q}\text{-}a.s. \quad x \mapsto X_t^x \quad \text{is non-decreasing}\ldots$$

- Hence

$$\mathbb{Q}\text{-}a.s. \quad x \mapsto f'(X_t^x) \quad \text{is non-decreasing}\ldots$$

- and so is

$$\partial_x \mathbb{E}\, f(X_T^x) = \mathbb{E}_{\mathbb{Q}}\, f'(X_T^x).$$

- Which ensures that $x \mapsto \mathbb{E}\, f(X_T^x)$ is convex. □

- Few comments:
  - ▷ Free extension for free to any convex function using right derivative f'_r.
  - ▷ Note that there is **no convexity assumption (!)** required on σ.
  - ▷ One step beyond: the present proof is one-dimensional. Any hope when $d \geq 2$ to switch from $f(X_T^x) \rightsquigarrow F\big((X_t^x)_{t\in[0,T]}\big)$?

[7] see Thm 3.7, chap. IX, Revuz-Yor, *Continuous martingales and Brownian motion*, Springer, 3rd ed. 1998

What about monotone convexity (in presence of a convex) drift) ?

- If f is smooth then

$$\partial_x \mathbb{E}\, f(X_T^x) = \mathbb{E}\Big[f'(X_T^x)\underbrace{e^{\int_0^T b'_x(s,X_s^x)ds}\, Y_T^{(x)}}_{\text{“new” tangent flow}}\Big] = \mathbb{E}_{\mathbb{Q}}\big[f'(X_T^x)e^{\int_0^T b'_x(s,X_s^x)ds}\big]$$

with $$X_t^x = x + \int_0^t (b+\sigma\sigma'_x)(s,X_s^x)ds + \int_0^t \sigma(s,X_s^x)dB_s.$$

- If f is convex non-decreasing and $b(t,\cdot)$ is convex in x then f' is non-negative and non-decreasing and $b'_x(t,\cdot)$ is non-decreasing. Hence

$$\partial_x \mathbb{E}\, f(X_T^x) \qquad \text{is non-negative non-decreasing}$$

i.e. $x \mapsto \mathbb{E}\, f(X_T^x)$ is is convex non-decreasing.

- **b convex requested but still not σ !**

A mouse hole ?

Figure: Who is the cat ? Who is the mouse ?

Smooth σ in $1D$ ($d = q = 1$): getting rid of convexity

- Assume $\sigma : \mathbb{R} \to \mathbb{R}_+$ $\mathcal{C}^2$, Lipschitz ($\|\sigma'\|_\infty < +\infty$).
- True Euler operator, $Z \sim \mathcal{N}(0,1)$:

$$Pf(x) = \mathbb{E}\, f\big(x + \sqrt{h}\sigma(x)Z\big).$$

- Assume w.l.g. (see later on) $f : \mathbb{R}^d \to \mathbb{R}$ $\mathcal{C}^2$ and convex, with bounded derivatives

$$\begin{aligned}(Pf)''(x) &= \mathbb{E}\big[f''\big(x+\sqrt{h}\sigma(x)Z\big)\big(1+\sqrt{h}\sigma'(x)Z\big)^2\big]\\ &\quad + \sqrt{h}\sigma''(x)\mathbb{E}\left[f'\big(x+\sqrt{h}\sigma(x)Z\big)Z\right]\\ &= \mathbb{E}\big[f''\big(x+\sqrt{h}\sigma(x)Z\big)\big(1+\sqrt{h}\sigma'(x)Z\big)^2\big]\\ &\quad + h\sigma\sigma''(x)\mathbb{E}\big[f''\big(x+\sqrt{h}\sigma(x)Z\big)\big] \qquad \text{[Stein I.P.: } \mathbb{E}g'(Z) = \mathbb{E}g(Z)Z\text{]}\\ &= \mathbb{E}\Big[f''\big(x+\sqrt{h}\sigma(x)Z\big)\underbrace{\Big(\big(1+\sqrt{h}\sigma'(x)Z\big)^2 + h\sigma\sigma''(x)\Big)}_{\text{always } \geq 0\ \forall\, Z(\omega)??}\Big].\end{aligned}$$

- No ! But... If we truncate : $Z \rightsquigarrow Z^h = Z\mathbf{1}_{\{|Z|\leq A_h\}}$, $Pf \rightsquigarrow \tilde{P}^h f$, then...

- Then, the same Stein-I.P. transform yields

$$(\tilde{P}^h f)''(x)$$
$$= \mathbb{E}\Big[f''(x+\sqrt{h}\sigma(x)Z^h)\underbrace{\big((1+\sqrt{h}\sigma'(x)Z^h)^2 + h\big(1-e^{-\frac{1}{2}(A_h^2-(Z^h)^2)}\big)\mathbf{1}_{\{Z^h\neq 0\}}\sigma\sigma''(x)\big)}_{\text{always } \geq 0\ \forall\, Z^h(\omega)??}\Big]$$

- YES !! If $A_h = A/\sqrt{h}$ with $A < \frac{1}{\|\sigma'\|_\infty}$ for $h = \frac{T}{n}$ small enough and

$$(\mathcal{S})\quad \sigma^2 \text{ semi-convex } (\exists \lambda \geq 0 \text{ s.t. } \sigma^2 + \lambda x^2 \text{ convex}) \tag{3}$$

- It clearly extends the $|\sigma|$ convex case !
- This semi-convexity property cannot be relaxed at the truncated Euler scheme level.

- So we have proved: for every convex $\mathcal{C}^2$-function f with bounded derivatives

$$x \mapsto \tilde{P}^h f(x) = \mathbb{E}\, f\big(x + \sqrt{h}\sigma(x) Z^h\big) \text{ is convex.}$$

- f Lipschitz continuous and convex can be approximated by convolution: let

$$f_\epsilon(x) = \mathbb{E}\, f(x + \epsilon\zeta),\ \zeta \sim \mathcal{N}(0,1).$$

- f_ϵ is convex, $\downarrow f$ as $\epsilon \downarrow 0$ and

$$f_\epsilon'(x) = \frac{1}{\epsilon}\mathbb{E}\big[(f(x+\epsilon\zeta) - f(x))\zeta\big] \text{ and } f_\epsilon''(x) = \frac{1}{\epsilon^2}\mathbb{E}\big[(f(x+\epsilon\zeta) - f(x))(\zeta^2 - 1)\big]$$

are both bounded.

- As $|f_\epsilon(x)| \le |f(x)| + \epsilon\, \mathbb{E}|\zeta|$,

$$\tilde{P}^h f = \lim_{\epsilon\to 0}{}^{\downarrow}\, \tilde{P} f_\epsilon \quad \text{so that} \quad \tilde{P}^h(f) \text{ is convex.}$$

- We still have that $(x, u) \mapsto \tilde{Q} f(x) = \mathbb{E}\, f(x + uZ^h)$ is convex and non-decreasing in u on $\mathbb{R}_+$ (see Jensen's inequality revisited!).

- Let consider the truncated Euler scheme $\widetilde{X}^h = \widetilde{X}^{(\sigma),h}$ associated with step $h = \frac{T}{n}$ (and $t_k^n = \frac{kT}{n}$), i.e.

$$\widetilde{X}^h_{t^n_{k+1}} = \widetilde{X}^h_{t^n_k} + \sigma(t^n_k, \widetilde{X}^h_{t^n_k})Z^h_{k+1}, \quad \widetilde{X}^h_0 = x$$

$$\text{with} \quad Z^h_{k+1} = \sqrt{\frac{n}{T}}\big(W_{t^n_{k+1}} - W_{t^n_k}\big)\mathbf{1}_{\{|W_{t^n_{k+1}} - W_{t^n_k}| \le A\}}.$$

- This scheme satisfies the convex propagation and ordering properties.
- Does it converge strongly in L^p toward to the diffusion $X^{(\sigma)}$? If "yes" then we proved:

  If $\sigma(t,\cdot)$ satisfies $(\mathcal{S})$ uniformly in $t \in [0,T]$ or $\theta(t,\cdot)$ satisfies $(\mathcal{S})$ uniformly in $t \in [0,T]$, if

$$0 \le \sigma \le \theta \quad \text{and} \quad X_0^{(\sigma)} \preceq_{cvx} X_0^{(\theta)} \Longrightarrow \forall\, t \in [0,T], \quad X_t^{(\sigma)} \preceq_{cvx} X_t^{(\theta)}$$

  and, when $\sigma(t,\cdot)$ satisfies $(\mathcal{S})$ uniformly in $t \in [0,T]$,

$$x \mapsto \mathbb{E}\, f(X_T^{(\sigma)}) \quad \text{is convex.}$$

- Extension to a new class of functionals, see later on (with B. Jourdain).

Proof of convergence of truncated Euler scheme

- Let $(\tilde{X}^h_{t_k})$ be the truncated Euler scheme with step $h = \frac{T}{n}$ i.e. implemented with $Z^h_k := Z_k \mathbf{1}_{\{|Z_k| \le A/\sqrt{h}\}}$, $(Z_k)_{k=1:n}$ i.i.d. $\mathcal{N}(0,1)$. Then, by independence,

$$\mathbb{P}(\widetilde{X}^h \neq \bar{X}^n) = \mathbb{P}\big(\exists\, k \in 1:n : |Z_k| \ge A/\sqrt{h}\big)$$
$$\le n\,\mathbb{P}(|Z| \ge A/\sqrt{h}).$$

- Using $\mathbb{P}(|Z| \ge x) \le e^{-\frac{x^2}{2}}$, $x > 0$, (and $h = \frac{T}{n}$)

$$\mathbb{P}(\widetilde{X}^h \neq \bar{X}^n) \le n\, e^{-\frac{An}{2T}} \to 0 \text{ as } n \to +\infty.$$

- As a consequence (...), if $X_0 \in L^{p'}(\mathbb{P})$, $p' > p$ (can be relaxed by a more direct (hence tedious) approach and an equivariance argument)

$$\Big\| \max_{k=0:n} \big|\widetilde{X}^h_{t_k} - \bar{X}^n_{t_k}\big| \Big\|_p \to 0 \quad \text{as} \quad n \to +\infty. \qquad \square$$

- The original proof can be adapted and finally the semi-convexity $(\mathcal{S})$ assumption can be relaxed in continuous time.

A 1-marginal $1D$ result

Theorem (Jourdain-P. 2023)

Let $\sigma, \theta \in Lip_{x,unif_t}([0,T]\times\mathbb{R},\mathbb{R})$. Let $X^{(\sigma)}$ and $X^{(\theta)}$ be the unique strong solutions to

$$dX_t^{(\sigma)} = \sigma(t, X_t^{(\sigma)})dW_t^{(\sigma)},\ X_0^{(\sigma)} \in L^1(\mathbb{P})$$
$$dX_t^{(\theta)} = \theta(t, X_t^{(\theta)})dW_t^{(\theta)},\ \ X_0^{(\theta)} \in L^1(\mathbb{P}),\ (W_t^{(\cdot)})_{t\in[0,T]} \text{ standard B.M.}$$

(a) *If $X_0^{(\sigma)} \preceq_{cvx} X_0^{(\theta)}$ and*

$$\begin{cases} (i)_\sigma & \sigma(t,.)^2 : \mathbb{R}\to\mathbb{R}_+ \text{ is semi-convex for every } t\in[0,T], \\ or & \\ (i)_\theta & \theta(t,.)^2 : \mathbb{R}\to\mathbb{R}_+ \text{ is semi-convex for every } t\in[0,T], \\ and & \\ (ii) & 0 \le \sigma(t,\cdot) \le \theta(t,\cdot) \text{ for every } t\in[0,T] \end{cases}$$

then:

– for every convex $f:\mathbb{R}\to\mathbb{R}$, $\mathbb{E}\, f(X_T^{(\sigma)}) \le \mathbb{E}\, f(X_T^{(\theta)}) \in (-\infty,+\infty]$

– if $(i)_\sigma$ holds true $x \mapsto \mathbb{E}\, f(X_T^{(\sigma),x})$ is convex.

- Slightly (technically) improves the result by [El karoui et al.] ($\simeq \sigma(t,\cdot)$ semi-convex).

A 1-marginal $1D$ result improved

Theorem (Jourdain-P. 2023)

Let $\sigma, \theta \in Lip_{x,unif_t}([0,T]\times\mathbb{R},\mathbb{R})$. Let $X^{(\sigma)}$ and $X^{(\theta)}$ be the unique strong solutions to

$$dX_t^{(\sigma)} = \sigma(t, X_t^{(\sigma)})dW_t^{(\sigma)},\ X_0^{(\sigma)} \in L^1(\mathbb{P})$$

$$dX_t^{(\theta)} = \theta(t, X_t^{(\theta)})dW_t^{(\theta)},\quad X_0^{(\theta)} \in L^1(\mathbb{P}),\ (W_t^{(\cdot)})_{t\in[0,T]} \text{ standard B.M.}$$

(a) If $X_0^{(\sigma)} \preceq_{cvx} X_0^{(\theta)}$ and

$$\begin{cases} (i)_\sigma & \varnothing \\ or & \\ (i)_\theta & \varnothing \\ and & \\ (ii) & 0 \le \sigma(t,\cdot) \le \theta(t,\cdot) \text{ for every } t\in[0,T] \end{cases}$$

then:

– for every convex $f:\mathbb{R}\to\mathbb{R}$, $\mathbb{E}\, f(X_T^{(\sigma)}) \le \mathbb{E}\, f(X_T^{(\theta)}) \in (-\infty,+\infty]$

– if $(i)_\sigma$ holds true $x \mapsto \mathbb{E}\, f(X_T^{(\sigma),x})$ is convex.

(b) It also works with diffusions sharing the same affine drift $b(t,x) = \alpha(t)x + \beta$.

- Significantly (technically) improves the result by [El karoui et al.] ($\simeq \sigma(t,\cdot)$

When the drift comes back into the game III (gentle reminder)

Theorem (Extended Hajek's Theorem, P. 2016, *Sém. Prob. XLVIII*)

Let $\sigma, \theta, b_1, b_2 \in \mathcal{C}_{lin_x, Unif_t}([0,T] \times \mathbb{R}, \mathbb{R})$. Let $X^{(\sigma)}$ and $X^{(\theta)}$ be the unique weak solutions to

$$\begin{aligned} dX_t^{(\sigma)} &= b_1(t, X_t^{(\sigma)})dt + \sigma(t, X_t^{(\sigma)})dW_t^{(\sigma)}, \; X_0^{(\sigma)} \in L^{1+\eta}(\mathbb{P}) \\ dX_t^{(\theta)} &= b_2(t, X_t^{(\theta)})dt + \theta(t, X_t^{(\theta)})dW_t^{(\theta)}, \; X_0^{(\theta)} \in L^{1+\eta}(\mathbb{P}), \end{aligned}$$

with $(W_t^{(\cdot)})_{t\in[0,T]}$ are a standard 1D B.M. If

$(i)_{b_1,\sigma}$ $b_1(t,.)$ and $\sigma(t,\cdot)$ convex, $t \in [0,T]$, *($\Rightarrow$ x-Lipschitz $Unif_t$)*

or

$(i)_{b_2,\theta}$ $b_2(t,.)$ and $\theta(t,\cdot)$ convex, $t \in [0,T]$.

Then, if $0 \le \sigma(t,\cdot) \le \theta(t,\cdot)$ and $X_0^{(\sigma)} \preceq_{icv} X_0^{(\theta)}$,

(a) For every $F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$, convex, pointwise non-decreasing with $\|\,.\,\|_{\sup}$-polynomial growth (hence $\|\,.\,\|_{\sup}$-continuous)

$$\mathbb{E}\, F(X^{(\sigma)}) \le \mathbb{E}\, F(X^{(\theta)}).$$

(b) If $(i)_{b_1,\sigma}$ holds then $x \mapsto \mathbb{E}\, F(X^{(\sigma)})$ is convex.

Back to non-decreasing convex order ($d = q = 1$): revisiting Hajek's theorem

- Assume $f : \mathbb{R} \to \mathbb{R}$ is smooth convex and non-decreasing.
- If
$$Pf(x) = \mathbb{E}\, f\big(x + hb(t,x) + \sqrt{h}\sigma(t,x)Z\big), \quad Z \sim \mathcal{N}(0,1)$$
with $b(t,\cdot)$ and $\sigma(t,\cdot)$ are uniformly Lipschitz then
$$(Pf)'(x) = \mathbb{E}\Big[\underbrace{f'\big(x + hb(t,x) + \sqrt{h}\sigma(t,x)Z\big)}_{\geq 0}\big(1 + hb'(t,x) + \sqrt{h}\sigma'_x(t,x)Z\big)\Big].$$
- Note that
$$1 + hb'(t,x) + \sqrt{h}\sigma'_x(t,x)Z \geq 1 - h\|b'_x\|_{\sup} - \sqrt{h}\|\sigma'_x\|_{\sup}|Z|.$$
- Hence, if $0 < h < (2\|b'_x\|_{\sup}\|)^{-1}$ then
$$1 + hb'(t,x) + \sqrt{h}\sigma'_x(t,x)Z \geq 0 \;\text{ on }\; \Big\{|Z| \leq \frac{1}{2\sqrt{h}\|\sigma'_x\|_{\sup}}\Big\}.$$
- Etc, like before (the two ideas can be combined...).

When the drift comes back into the game III

Theorem (Extended Hajek's Theorem, Jourdain-P. 2023)

Let $\sigma, \theta, b_1, b_2 \in \mathcal{C}_{lin_x}([0,T]\times\mathbb{R},\mathbb{R})$. *Let* $X^{(\sigma)}$ *and* $X^{(\theta)}$ *be the unique weak solutions to*

$$\begin{aligned} dX_t^{(\sigma)} &= b_1(t,X_t^{(\sigma)})dt + \sigma(t,X_t^{(\sigma)})dW_t^{(\sigma)},\ X_0^{(\sigma)} \in L^{1+\eta}(\mathbb{P}) \\ dX_t^{(\theta)} &= b_2(t,X_t^{(\theta)})dt + \theta(t,X_t^{(\theta)})dW_t^{(\theta)},\ X_0^{(\theta)} \in L^{1+\eta}(\mathbb{P}), \end{aligned}$$

with $(W_t^{(\cdot)})_{t\in[0,T]}$ *are a standard 1D B.M. If*

$(i)_{b_1,\sigma}$ $b_1(t,.)$ *convex,* $t \in [0,T]$, *and* $\varnothing$

or

$(i)_{b_2,\theta}$ $b_2(t,.)$ *convex ,* $t \in [0,T]$ *and* $\varnothing$.

Then, if $0 \le \sigma(t,\cdot) \le \theta(t,\cdot)$ *and* $X_0^{(\sigma)} \preceq_{icv} X_0^{(\theta)}$,

(a) *For every* $F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$, *convex, pointwise non-decreasing with* $\|\,.\,\|_{\sup}$*-polynomial growth (hence* $\|\,.\,\|_{\sup}$*-continuous)*

$$\mathbb{E}\,F(X^{(\sigma)}) \le \mathbb{E}\,F(X^{(\theta)}).$$

(b) *If* $(i)_{b_1,\sigma}$ *holds then* $x \mapsto \mathbb{E}\,F(X^{(\sigma)})$ *is convex.*

A first conclusion and provisional remarks on $1D$ setting

- Relaxing convexity in x of the diffusion coefficient $\sigma(t, x)$ can be seen as a second extension of Hajek's theorem (for diffusions with no drift).
- This result is deeply one dimensional and cannot be extended to higher dimension at a reasonable level of generality (to our best knowledge).
- The second results for marginal increasing convex ordering for diffusions having convex drifts "$b^\sigma \leq b^\theta$" is essentially Hajek's.
- A combination of the two truncations is possible (in progress with B. Jourdain) and would be a first strict improvement of Hajek's theorem.
- Applications to local volatility models (like CEV) extending results by El Karoui-Jeanblanc-Shreve to continuous time path-dependent options.

Functional extension ? Directionally convex functionals

- A function $f:\mathbb{R}^d\to\mathbb{R}$ is *directionally convex* if
  - $\forall\, i,\ x_i\mapsto f(x_1,\ldots,x_i,\ldots x_d)$ is convex
  - $\forall\, j,\ x_j\mapsto \partial_{x_i} f(x_1,\ldots,x_i,\ldots x_d)$ is non-decreasing.

  or, equivalently, f Is Borel measurable and

$$\forall\, x\in\mathbb{R}^m,\ \forall\, y,z\in\mathbb{R}^m_+,\ f(x+y+z)-f(x+y)-f(x+z)+f(x)\geq 0.$$

- A functional $F:\mathcal{C}([0,T],\mathbb{R})\to\mathbb{R}$ is *directionally convex* if it is measurable and

$$\forall x\in\mathcal{C}([0,T],\mathbb{R}),\ \forall\, y,z\in\mathcal{C}([0,T],\mathbb{R}_+),$$
$$F(x+y+z)-F(x+y)-F(x+z)+F(x)\geq 0.$$

Theorem

The 1D version of both functional comparison-propagation theorems remains true (under standard Lipschitz but without convexity assumptions) for the class of continuous directionally convex functionals on $C([0,T],\mathbb{R})$ with r-polynomial growth if $X_0^{(\sigma)}$ and $X_0^{(\vartheta)}\in L^r(\mathbb{P})$.

Examples I ($\mathbb{R}^d$)

Convexité, convexité marginale et convexité directionnelle

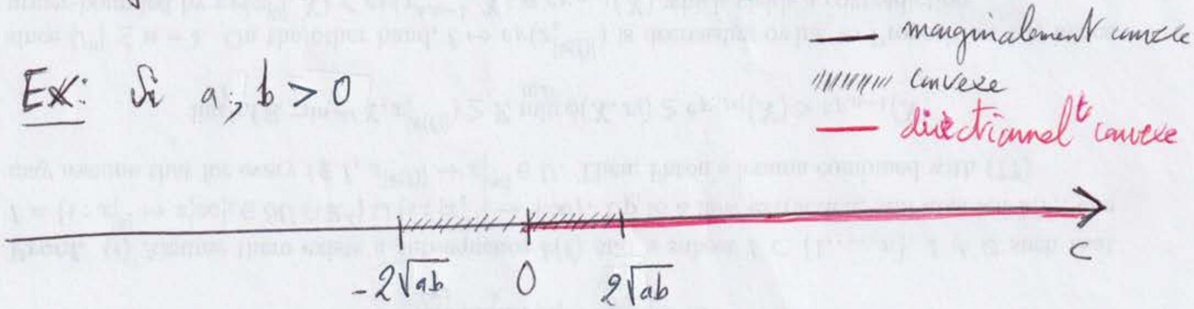
On considère la fonction

$$f(x,y) = \frac{1}{2}\left(ax^2+by^2+cxy\right)$$

- f marginalement convexe ssi $a, b \geq 0$
- f convexe ssi $D^2 f(x,y) \in \mathcal{S}^+(d,\mathbb{R}) \iff \begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix} \in \mathcal{S}^+(d,\mathbb{R})$

  $\iff a, b \geq 0$ et $c^2 \leq 4ab$
- f directionnellement convexe ssi $a, b, c \geq 0$

Ex: Si $a, b > 0$



Examples II (functional)

- Smooth directionally convex functionals: if $F : C([0,T],\mathbb{R}) \to \mathbb{R}$ is $\mathcal{C}^2$, F is directionally convex iff

$$\forall x, u,\ v \in C([0,T],\mathbb{R}), \quad u,\ v \geq 0 \Longrightarrow D^2F(x).(u,v) \geq 0$$

- Let φ, $\Phi : \mathbb{R} \to \mathbb{R}$,

$$\forall x, \in C([0,T],\mathbb{R}), \quad F(x) = \Phi\left(\int_0^T \varphi(x(s))ds\right).$$

  - F is convex iff φ is convex and Φ is non-decreasing convex.
  - F is directionally convex iff both φ and Φ are non-decreasing convex.

Extensions

This provides as systematic approach which successfully works with

- Jump diffusions,
- (Possibly) path-dependent American style options, (*Sém. Proba XLVIII*, 2016),
- BSDE (unfortunately without "Z" in the driver),
- McKean-Vlasov SDEs (with applications to MFG, with Y. Liu (*AAP*, 2023),
- Volterra equations (with application to rough volatility modeling, with B. Jourdain (Fin. & Stoch., 2024)),
- Stochastic control (a first on going work with C. Yeo) with application to swing option on gas
- ...

Let's have a short look...

The case of jump diffusions

▷ Lévy process: Let $Z = (Z_t)_{t\in[0,T]}$ be a Lévy process with Lévy measure ν satisfying

- $\displaystyle\int_{0<|z|\leq 1} |z|^2\nu(dz) < +\infty$ of course...
- $\displaystyle\int_{|z|\geq 1} |z|^p\nu(dz) < +\infty,\ p \in [1, +\infty)$ (hence $Z_t \in L^1(\mathbb{P}),\ t \in [0, T]$).
- $\mathbb{E}\, Z_1 = 0$.

Then

$$(Z_t)_{t\in[0,T]} \text{ is an centered } \mathcal{F}^Z\text{-martingale.}$$

Theorem (P. 2016, *Séminaire de Proba XLVIII* , $d = q = 1$, "weak version", not yet updated $d, q \geq 1$ but in progress)

Let $\kappa_i \in \mathcal{C}_{lin_x, unif_t}([0, T] \times \mathbb{R})$, $i = 1, 2$, be continuous functions Let $X^{(\kappa_i)} = (X_t^{(\kappa_i)})_{t\in[0,T]}$ be the diffusion processes, unique weak solutions to

$$dX_t^{(\kappa_i)} = \kappa_i(t, X_{t_-}^{(\kappa_i)})dZ_t,\ X_0^{(\kappa_i)} \in L^p(\mathbb{P}),\ i = 1, 2.$$

(a) Z_1 centered: Assume $\kappa = \kappa_1$ or κ_2 satisfies: $\forall\, t \in [0, T]$, $\kappa(t, .)$ convex and that

$$0 \leq \kappa_1 \leq \kappa_2.$$

(b) Z_1 radial: If $Z_1 \stackrel{\mathcal{L}}{=} -Z_1$, $|\kappa|$ is convex in x and κ_i satisfy

$$|\kappa_1| \leq |\kappa_2|.$$

(i) Let $F : \mathbb{D}([0, T], \mathbb{R}) \to \mathbb{R}$ be a convex Skorokhod-continuous functional with r-polynomial growth, $r < p$

$$\forall\, \alpha \in \mathbb{D}([0, T], \mathbb{R}), \quad |F(\alpha)| \leq C(1 + \|\alpha\|_{\sup}^r),\ 0 < r < p.$$

(ii) If $\kappa_1(t, \cdot)$ convex, $t \in [0, T]$, then for everty F as above

$$x \mapsto \mathbb{E}\, F(X^{(\kappa_1),x}) \ \text{ is convex.}$$

Key argument when $d = q = 1$

- Discrete time approach is similar to Brownian diffusions
- Transfer phase is based on the Skorokhod functional weak convergence of the Euler scheme toward the martingale jump diffusion.
- Which in turn relies on functional weak convergence of stochastic integrals (see e.g. [Mémin-Jakubowski-P., *PTRF*, 1989]).
- A "strong" version with Lipschitz coefficients κ_i (uniformly in t) should work, possible without Skorokhod topology.
- Higher dimensions should work too if Z is radial (but not yet proved to our best knowledge).

Discrete time optimal stopping (Bermuda options). . .

. . . of ARCH models in 1-dimension.

▷ Dynamics: Still. . . $(Z_k)_{1\le k\le n}$ be a sequence of independent, (centered and) symmetric r.v.

$$\begin{aligned} X_{k+1} &= X_k + \sigma_k(X_k)\, Z_{k+1},\; X_0 \in L^1(\mathbb{P}) \\ Y_{k+1} &= Y_k + \theta_k(Y_k)\, Z_{k+1}, \quad 0 \le k \le n-1,\; Y_0 \in L^1(\mathbb{P}) \end{aligned}$$

where $\sigma_k,\, \theta_k : \mathbb{R} \to \mathbb{R},\; k = 0, \ldots, n-1$ with (at most) linear growth.

Snell envelopes and Bermuda/American options

▷ Let $F_k : \mathbb{R}^{k+1} \to \mathbb{R}_+$, $k = 0, \dots, n$ be a sequence of non-negative *convex* (payoff) functions with r-polynomial growth for the sup norm.

▷ Let $\mathcal{F} = (\mathcal{F}_k)_{0\le k\le n}$ be a filtration such that Z_k is $\mathcal{F}_k$-adapted and Z_k is independent of $\mathcal{F}_{k-1}$, $k = 1, \dots n$.

▷ Snell envelopes of the reward processes $\big(F_k(X_{0:k})\big)_{0\le k\le n}$ and $\big(F_k(Y_{0:k})\big)_{0\le k\le n}$

$$U_k = \mathbb{P}\text{-esssup}\Big\{\mathbb{E}\big(F_\tau(X_{0:\tau})\,|\,\mathcal{F}_k\big),\ \tau\ \mathcal{F}\text{-stopping time}, \tau \ge k\Big\}$$

and

$$V_k = \mathbb{P}\text{-esssup}\Big\{\mathbb{E}\big(F_\tau(Y_{0:\tau})\,|\,\mathcal{F}_k\big),\ \tau\ \mathcal{F}\text{-stopping time}, \tau \ge k\Big\}.$$

▷ These are the lowest super-martingales that dominate the reward processes.

Backward Dynamic programming Principle

Proposition (Backward Dynamic programming Principle (*BDDP*))

(*a*) *The Snell envelope satisfies*

$$U_n = F_n(X_{0:n}), \qquad U_k = \max\left(F_k(X_{0:k}), \mathbb{E}\left(U_{k+1} \,|\, \mathcal{F}_k\right)\right),\ k = 0 : n-1.$$

(*b*) *One has*

$$U_k = u_k(X_{0:k}) \quad \mathbb{P}\text{-}a.s.,\ k = 0, \ldots, n-1,$$

where the functions $u_k : \mathbb{R}^{k+1} \to \mathbb{R}_+$, $k = 0 : n$, *satisfy the functional* BDDP

$$u_n(x_{0;n}) = F_n(x_{0;n}), \quad u_k(x_{0:k}) = \max\Big(F_k(x_{0:k}), Q_{k+1}u_{k+1}(x_{0:k}, x_k + .))(\sigma_k(x_k))\Big)$$
$$k = 0, \ldots, n-1.$$

- Propagation of the convexity: Note that $(a,b) \mapsto \max(a,b)$ is non-decreasing in a and b and "copy-paste" the proofs for a fixed functional using the "revisited" Jensen's Inequality.

Proposition

(a) Convex ordering. If, either

$$\begin{cases} (*)_\sigma \quad |\sigma_k| \text{ is convex for every } k = 0 : n-1 \\ \textit{or} \\ (*)_\theta \quad |\theta_k| \text{ is convex for every } k = 0 : n-1 \end{cases}$$

and

$$|\sigma_k| \le |\theta_k|,\ k = 0, \ldots, n-1$$

then,

$$u_k(x_{0:k}) \le v_k(x_{0:k}),\ k = 0, \ldots, n.$$

(b) Convexity. If $(*)_\sigma$ holds then

$$x \longmapsto u_k(x_{0:k}) \text{ is a convex function on } \mathbb{R}^{k+1}.$$

In particular, if $X_0 \preceq_{cvx} Y_0$ then $\mathbb{E}\, U_0 = \mathbb{E}\, u_0(X_0) \le \mathbb{E}\, u_0(Y_0) \le \mathbb{E}\, v_0(Y_0) = \mathbb{E}\, V_0$.

. . .

▷ Idem for $v_k : \mathbb{R}^{k+1} \to \mathbb{R}$ in connection with the $(\mathbb{P}, \mathcal{F})$-Snell envelope V.

▷ Note that u_{k+1} convex still implies

$$\xi \longmapsto \big(Q_{k+1} u_{k+1}(x_{0:k}, \cdot)\big)(x_k, \xi) \quad \text{is non-decreasing on } \mathbb{R}_+.$$

▷ Comparison Principle ($|\sigma_k| \le |\theta_k|$): Backward induction to prove $u_k \le v_k$, $k = 0 : n$ (obvious if $k = n$).

Assume $u_{k+1} \le v_{k+1}$, $k+1 \le n$. For every $x_{0:k} \in \mathbb{R}^{k+1}$

$$\begin{aligned} u_k(x_{0:k}) &\le \max\Big(F_k(x_{0:k}), \big(Q_{k+1} u_{k+1}(x_{0:k}, \cdot)\big)(x_k, \theta_k(x_k))\Big) \\ &\le \max\Big(F_k((x_{0:k}), \big(Q_{k+1} v_{k+1}(x_{0:k}, \cdot)\big)(x_k, \theta_k(x_k))\Big) = v_k(x_{0:k}). \end{aligned}$$

If $k = 0$, we get

$$\mathbb{E}\, U_0 = u_0(x) \le v_0(x) = \mathbb{E}\, V_0. \qquad \square$$

Back to continuous time

▷ Let $F : [0, T] \times \mathcal{C}([0, T], \mathbb{R}^d) \to \mathbb{R}_+$ be a Lipschitz continuous functional and the resulting American payoffs processes $\big(F(t, (X^{(\sigma)})^t)\big)_{t\in[0,T]}$ and $\big(F(t, (Y^{(\theta)})^t)\big)_{t\in[0,T]}$.

▷ Snell envelopes of the Euler schemes of martingale diffusions X and Y

$$U^{(n)} = \mathbb{P}\text{-Snell}\big(F_k(\bar{X}^{(\sigma),n}_{0:k})_{k=0:n}\big) \quad V^{(n)} = \mathbb{P}\text{-Snell}\big(F_k(\bar{Y}^{(\theta),n}_{0:k})_{k=0:n}\big).$$

▷ Convergence: In the case of Brownian diffusions, it is a classical result (with convergence rates in fact, see $e.g.$ ([8]) that

$$\Big\| \max_{0\le k\le n} |U^{(n)}_k - U^X_{t^n_k}| \Big\|_p \to 0 \;\text{ and }\; \Big\| \max_{0\le k\le n} |V^{(n)}_k - V^Y_{t^n_k}| \Big\|_p \to 0 \text{ as } n \to +\infty$$

▷ Etc (limit theorems).

[8] V. Bally-P. ('03), Error analysis of the quantization algorithm for obstacle problems, *Stochastic Processes & Their Applications*, 106(1), 1-40, 2003

▷ Conclusion: As usual...

Theorem (P. 2016)

Under former partitioning or dominating convexity assumptions on $\sigma(t,\cdot)$ *and* $\theta(t,\cdot)$, $F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}_+$ *convex and continuous and* $X_0^{(\sigma)} \preceq X_0^{(\theta)}$ *one has*

$$\mathbb{E}\, U_0^{X^{(\sigma)}} \le \mathbb{E}\, V_0^{X^{(\theta)}}$$

and, if $\sigma(t,\cdot)$ *is convex,* $x \mapsto u_0(x) := \mathbb{E}\, U_0^{X^{(\sigma),x}}$ *is convex.*

Warning! No standard “réduites” at time $t > 0$, due to path-dependence. If $F(t,x) = h\big(t, x(t)\big)$, then $u_t(x) \le v_t(x)$ for every $t \in [0,T]$ if $h(t,\cdot)$ is convex for every t and h*Lispchitz*.

Jump martingale diffusions: what makes problem?

▷ Discrete time step: Identical.

▷ From discrete to continuous time: Still the Euler scheme. But we have to make the Snell envelopes converge. . . How to proceed?

Filtration enlargement argument/trick

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration and let Y be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted càdlàg process defined on a probability space $(\Omega,\mathcal{A},\mathbb{P})$ so that

$$\forall\, t \in [0,T], \quad \mathcal{F}_t^Y \subset \mathcal{F}_t$$

We introduce the so-called $\mathcal{H}$-assumption (on the filtration $(\mathcal{F}_t)_{t\in[0,T]}$):

$$(\mathcal{H}) \equiv \forall\, H \in \mathcal{F}_T^Y,\ \text{bounded},\ \mathbb{E}\big(H\,|\,\mathcal{F}_t\big) = \mathbb{E}\big(H\,|\,\mathcal{F}_t^Y\big)\ \mathbb{P}\text{-}a.s.$$

Example: $\mathcal{F}_t = \sigma(\mathcal{F}_t^Y, \Xi)$, $\Xi \perp\!\!\!\perp Y$.

Theorem (Lamberton-P., 1990)

([a]) $\triangleright$ *Let $(X^n)_{n\geq 1}$ be a sequence of quasi-left càdlàg processes defined on a probability spaces $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ of (D)-class and satisfying the Aldous criterion. Let $(\tau_n^*)_{n\geq 1}$ be a sequence of $(\mathcal{F}^{X^n}, \mathbb{P}^n)$-optimal stopping times. If $(X^n)_{n\geq 1}$ is uniformly integrable and satisfies*

$$X^n \stackrel{\mathcal{L}(Skor)}{\longrightarrow} X, \; \mathbb{P}_X = \mathbb{P} \textit{ probability measure on } (\mathbb{D}([0,T],\mathbb{R}), \mathcal{D}_T).$$

$\triangleright$ *Non-degeneracy of $(\tau_n^*)_{n\geq 1}$: every limiting value $\mathbb{Q}$ of $\mathcal{L}(X^n, \tau_n^*)$ on $\mathbb{D}([0,T],\mathbb{R}) \times [0,T]$ satisfies the $(\mathcal{H})$ property [...], then*

$$\lim_n \mathbb{E}_{\mathbb{P}^n} U_0^{X^n} = \mathbb{E}_{\mathbb{P}} U_0^X.$$

$\triangleright$ *If the optimal stopping problem related to $(X, \mathbb{Q}, \mathcal{D}^\theta)$ has a unique solution in distribution, say $\mu^*_{\tau^*}$, not depending on $\mathbb{Q}$, then $\tau_n^* \stackrel{[0,T]}{\longrightarrow} \mu^*_{\tau^*}$.*

[a] Sur l'approximation des réduites, *Annales IHP B*, 1990.

Theorem (P. 2016)

Under the usual assumptions on κ_i, $i = 1, 2$, the Lévy process $(Z_t)_{t\in[0,T]}$ (through Z_1) and the American payoffs $(F_t)_{t\in[0,T]}$ (convexity and polynomial growth) and the ordering of the starting values of the SDEs, then the Snell envelopes at time 0 associated to (F_t) and the jump diffusions $X^{(\kappa_i),x}$, $i = 1, 2$, satisfy

$$\mathbb{E}\, U_0^{(1)} \leq \mathbb{E}\, V_0^{(1)}.$$

In particular the resulting "réduites" (when both diffusions start from x) satisfy

$$u_0^{(\kappa_1)}(x) \leq u_0^{(\kappa_2)}(x)$$

Moreover, if $\kappa_1(t, \cdot)$ si convex for every $t \in [0, T]$, then $x \mapsto u_t^{(\kappa_1)}(x)$ is convex.

All the efforts are focused on showing that the filtration enlargement assumption $(\mathcal{H})$ is satisfied by any limiting distribution $\mathbb{Q}$.